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Variation of the Canonical Height on Elliptic Surfaces III: Global Boundedness Properties

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Let $E \to C$ be an elliptic surface defined over a number field K, let $P: C \to E$ be a section, and for each $t \in C(K)$, let $\hat{h}(P_t)$ be the canonical height of $P_t \in E_t(\bar{K})$. Tate has used a global argument to show that, up to a bounded quantity, the function $t \mapsto \hat{h}(P_t)$ is equal to a Weil height function $h_C(t)$ on C. In this paper we precisely describe the behavior of the difference $\hat{h}(P_t) - h_C(t)$ as a function of t.
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Let $E \to C$ be an elliptic surface defined over a number field K, and let $P: C \to E$ be a non-zero section also defined over K. For each point $t \in C(\bar{K})$ one obtains an algebraic point P_t on the fiber $E_t(\bar{K})$ and it is natural to ask how the canonical height $\hat{h}_{E_t}(P_t)$ varies as a function of t. A theorem of Tate [10] says that there is a divisor class $\hat{\eta}_E(P) \in \mathrm{Div}(C) \otimes \mathbb{Q}$ such that

$$\hat{h}(P_t) = h_{\hat{\eta}_E(P)}(t) + O_P(1) \qquad \text{for all} \quad t \in C(\bar{K}). \tag{1}$$

For a discussion of the history of this problem, we refer the reader to the introduction of [8].

In the first of this series of papers [8] we showed for some particular examples that the variation of $\hat{h}(P_t)$ is actually much more regular than predicted by (1). The proofs in [8] relied on the decomposition of the canonical height into a sum of local heights

$$\hat{h}_{E_t}(P_t) = \sum_v \hat{\lambda}_{E_t}(P_t; v),$$

one for each absolute value on K. In the second paper [9] we showed that, for any point $t_0 \in C$, the difference of local heights

$$\hat{\lambda}_{E_t}(P_t; v) - \hat{\lambda}_E(P; \mathrm{ord}_{t_0})\, \lambda_{C,(t_0)}(t; v) \tag{2}$$

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extends to an almost v-adic analytic function in a v-adic neighborhood of t_0. In this paper we are going to prove that for almost all absolute values v on K, the difference (2) is identically zero. Combining this with our previous result [9], we will deduce a very general result which explains quite precisely how the $O_P(1)$ in Tate's formula (1) varies as a function of t.

In order to precisely state our first main result, we need to set some notation which will remain fixed throughout the remainder of this paper.

$K/\mathbb{Q}$	a number field.
C/K	a smooth projective curve of genus g.
$\lambda_{C,D}$	a Weil local height on C relative to the divisor $D \in \mathrm{Div}(C) \otimes \mathbb{Q}$. See [5, Chap. 10].
$E \to C$	an elliptic surface defined over K with zero section $\mathcal{O}: C \to E$.
P	a non-zero section $P: C \to E$.
$\hat{\lambda}_E(P; \mathrm{ord}_\gamma)$	the canonical local height of the section P at the valuation ord_γ, where $\gamma \in C$.
$\hat{\eta}_E(P)$	the canonical height divisor of the section P, defined by $$\hat{\eta}_E(P) = \sum_{\gamma \in C} \hat{\lambda}_E(P; \mathrm{ord}_\gamma) \cdot (\gamma) \in \mathrm{Div}(C) \otimes \mathbb{Q}.$$
$\hat{\lambda}_{E_t}(\cdot\,; v)$	for each $t \in C$ such that the fiber E_t is smooth, the Néron canonical local height function for the absolute value v, $$\hat{\lambda}_{E_t}(\cdot\,; v): (E_t \backslash \mathcal{O}_t) \to \mathbb{R}.$$

THEOREM III.0.1. *With notation as above, there is a finite set of places $S \subset M_K$ so that for all $v \in M \backslash S$ and all $t \in C \backslash |\hat{\eta}_E(P)|$,*

$$\hat{\lambda}_{E_t}(P_t; v) - \lambda_{C, \hat{\eta}_E(P)}(t; v) = 0.$$

Note that the set of places S will depend on the particular (normalized) Weil height function $\lambda_{C,\hat{\eta}_E(P)}$ that we choose.

Remark. Call [1] has proven Theorem III.0.1 in the case that the section P goes through the identity component of every fiber of E. Call's proof, which is based on Tate's telescoping sum method, is quite different from the one we will give.

Combining Theorem III.0.1 with our earlier local result [9, Theorem II.0.1], we are able to give the following description of the variation of the canonical height of P_t for varying t.

THEOREM III.0.2. *With notation as above, there is a finite set of places $S \subset M_K$ and a function $F_P \in \mathscr{F}_{E,S}$ such that*

$$\hat{h}_{E_t}(P_t) = h_{C,\bar{\eta}_E(P)}(t) + F_P(t) \qquad \text{for all} \quad t \in C(\bar{K}) \text{ with } E_t \text{ smooth and } P_t \neq \mathcal{O}_t.$$

Here $h_{C,\bar{\eta}_E(P)}$ is an analytic Weil height function as defined in Section 3, and $\mathscr{F}_{E,S}$ is the collection of functions $C(\bar{K}) \to \mathbb{R}$ also described in Section 3.

We will also prove a local version of Theorem III.0.2 which roughly says that

$$\hat{\lambda}_{E_t}(P_t; v) = \lambda_{C,\bar{\eta}_E(P)}(t) + F_v(t) \qquad \text{for all} \quad t \in C(\mathbb{C}_v) \text{ with } E_t \text{ smooth and } P_t \neq \mathcal{O}_t.$$

For the complete statement and proof, see Theorem III.4.1.

Although we will delay giving a precise description of the set $\mathscr{F}_{E,S}$ until Section 3, we will briefly indicate here what the functions in this set look like. For each $v \in S$, fix a function $\phi_v\colon C(\mathbb{C}_v) \to \mathbb{R}$. For each $w \in M$ extending v we take an embedding $\bar{K} \hookrightarrow \bar{K}_w \hookrightarrow \mathbb{C}_v$ which we denote by $t \to t_w$. Then for $t \in C(\bar{K})$, say $t \in C(L)$ for a finite extension L/K, we define $\phi(t)$ by

$$\phi(t) = \sum_{v \in S} \sum_{\substack{w \in M_L \\ w \mid v}} \frac{[L_w : K_v]}{[L : K]} \phi_v(t_w).$$

These are the sorts of functions that are in $\mathscr{F}_{E,S}$, where the ϕ_v's are also required to have various nice properties. For example, let $C_0 \subset C$ be the open set over which E_t is smooth. Then for archimedean v the ϕ_v's are real analytic on $C_0(\mathbb{C}_v)$, and for non-archimedean v the ϕ_v's are locally constant on $C_0(\mathbb{C}_v)$. The description of the ϕ_v's near points of $C \backslash C_0$ is more complicated.

To further assist the reader in understanding the import of Theorem III.0.2., we give the following special case.

COROLLARY III.0.3. *Let $E \to \mathbb{P}^1$ be an elliptic surface defined over $\mathbb{Q}$ and let $P\colon \mathbb{P}^1 \to E$ be a non-zero section also defined over $\mathbb{Q}$. There is a finite set of places $S \subset M_{\mathbb{Q}}$ such that for any point*

$$\gamma = (\gamma_p) \in \prod_{p \in S} \mathbb{P}^1(\mathbb{Q}_p) \tag{3}$$

there is a neighborhood $U_\gamma \subset_{p \in S} \prod \mathbb{P}^1(\mathbb{Q}_p)$ *of* γ *so that the following are true*:

(a) *Suppose that the fibers* E_{γ_p} *are smooth for all* $p \in S$. *Then there is a real-analytic function* $\phi\colon \operatorname{proj}_{\mathbb{R}}(U_\gamma) \to \mathbb{R}$ *such that*

$$\hat{h}_{E_t}(P_t) = h_{\mathbb{P}^1, \hat{\eta}_E(P)}(t) + \phi(t) \qquad \text{for all} \quad t \in U_\gamma \cap \mathbb{P}^1(\mathbb{Q}) \text{ with } P_t \neq \mathcal{O}_t.$$

Here $\operatorname{proj}_{\mathbb{R}}(U_\gamma)$ *is the projection of* U_γ *to* $\mathbb{P}^1(\mathbb{R})$, *and we embed* $\mathbb{P}^1(\mathbb{Q})$ *into* $\prod \mathbb{P}^1(\mathbb{Q}_p)$ *in the natural way.*

(b) *In general, for each* $p \in S$ *let* u_p *be a uniformizer at* γ_p. *Then there is a constant* κ *so that*

$$\hat{h}_{E_t}(P_t) = h_{\mathbb{P}^1, \hat{\eta}_E(P)}(t) + \kappa + \sum_{p \in S} O\left(\frac{1}{\log|u_p(t)|_p}\right) \qquad \text{for all} \quad t \in U_\gamma \cap \mathbb{P}^1(\mathbb{Q}).$$

Remark. Note that the product (3) is compact, since S is a finite set of places. For each point γ in the product, Corollary III.0.3 gives an open set U_γ, so by compactness we can cover the product by a finite number of such open sets. In particular, there are a finite number of estimates as described in Corollary III.0.3(b) such that one of them holds for every point in $\mathbb{P}^1(\mathbb{Q})$.

In order to illustrate Theorem III.0.2 and Corollary III.0.3, we return to the examples considered in [8]. Let $E \to \mathbb{P}^1$ and $P\colon \mathbb{P}^1 \to E$ be the elliptic surface and section

$$E: Y^2 = X^3 + T^2(1 - T^2)X, \qquad P = (T^2, T^2).$$

We are going to look in a neighborhood of $T = \infty$, so it is convenient to let $T = 1/t$ and make the change of variables $(x, y) = (t^2X, t^3Y)$. This gives the equation

$$E: y^2 = x^3 + (t^2 - 1)x, \qquad P = (1, t).$$

Note that the fiber E_0 (i.e., over $t = 0$) is smooth. Applying Corollary III.0.3(a) with $\gamma_p = 0$ for all p, we find a function $\phi(t) \in \mathbb{R}[\![t]\!]$ which is real analytic in an ε-neighborhood of 0 and an integer $N \geqslant 1$ so that

$$\hat{h}_{E_t}(P_t) = h_{\mathbb{P}^1, \hat{\eta}_E(P)}(t) + \phi(t) \qquad \text{for} \quad t \in \mathbb{Q} \text{ with } |t| < \varepsilon \text{ and } t \equiv 0 \ (\mathrm{mod}\ N). \tag{4}$$

It is easy enough to calculate $\hat{\eta}_E(P)$. Using the methods in [3] or [7], we find

$$\hat{\eta}_E(P)=\tfrac{1}{4}(1)+\tfrac{1}{4}(-1)\in \operatorname{Div}(\mathbb{P}^1)\otimes \mathbb{Q}.$$

Here (1) means the point with $t=1$, and similarly for (-1). Now, if $\tau\in\mathbb{Q}$ is any point, then an analytic Weil local height for the divisor (τ) is given by

$$\hat{\lambda}_{\mathbb{P}^1,(\tau)}(t;p)=\begin{cases}\log^+|t-\tau|_p^{-1} & \text{if } p\neq\infty,\\ \tfrac{1}{2}\log(1+|t-\tau|_\infty^{-2}) & \text{if } p=\infty.\end{cases}$$

(See Example 1 in Section 3). In particular, if we write $t=a/b\in\mathbb{Q}$ and $\tau=\alpha/\beta\in\mathbb{Q}$ and assume that $\gcd(b,\beta)=1$, then

$$\begin{aligned}h_{\mathbb{P}^1,(\tau)}(t)&=\sum_{p\neq\infty}\log^+\left|\frac{a}{b}-\frac{\alpha}{\beta}\right|_p^{-1}+\frac{1}{2}\log\left(1+\left|\frac{a}{b}-\frac{\alpha}{\beta}\right|_\infty^{-2}\right)\\&=\log|a\beta-b\alpha|_\infty+\frac{1}{2}\log\left(1+\left|\frac{b\beta}{a\beta-b\alpha}\right|_\infty^{2}\right)\\&=\frac{1}{2}\log(|a\beta-b\alpha|_\infty^2+|b\beta|_\infty^2).\end{aligned}$$

Applying this with $\tau=1$ and $\tau=-1$, we find that

$$\begin{aligned}h_{\mathbb{P}^1,\hat{\eta}_E(P)}(t)&=h_{\mathbb{P}^1,\hat{\eta}_E(P)}\left(\frac{a}{b}\right)\\&=\frac{1}{8}\log(|a-b|^2+b^2)+\frac{1}{8}\log(|a+b|^2+b^2)\\&=\frac{1}{2}\log|b|+\frac{1}{8}\log(|t-1|^2+1)+\frac{1}{8}\log(|t+1|^2+1)\\&=\frac{1}{2}\log|b|+\frac{1}{8}\log(t^4+4).\end{aligned}$$

(We have dropped the ∞ subscript for the archimedean absolute value on $\mathbb{Q}$.)

This says that

$$h_{\mathbb{P}^1,\hat{\eta}_E(P)}(t)=\frac{1}{2}\log|b|+\phi_1(t),$$

where $\phi_1\in\mathbb{R}[\![t]\!]$ is real analytic at $t=0$. Now (4) gives

$$\hat{h}_{E_t}(P_t)=\frac{1}{2}\log|b|+\phi_2(t)$$

$$\text{for}\quad t=\frac{a}{b}\in\mathbb{Q}\text{ with }|t|\ll 1\text{ and }t\equiv 0\ (\mathrm{mod}\ N). \tag{5}$$

How does this compare with what we proved in [8]? Combining Propositions I.2.1 and I.3.1 of [8] (see also the proof of I.4.1 and note that our t is the reciprocal of the one used in [8]), we find that

$$\hat{h}_{E_t}(P_t) = \tfrac{1}{2}\log|b| + \phi(t^2) + \begin{cases} \tfrac{1}{4}\log 2 & \text{if } \operatorname{ord}_2(t) \leqslant 0, \\ 0 & \text{if } \operatorname{ord}_2(t) > 0. \end{cases}$$

Here $\phi \in \mathbb{R}[\![u]\!]$ is real analytic at 0. So we get (5) with the additional information that $N=2$. (The case with $1/t \in \mathbb{Z}$ is covered in [8, I.0.1].) This shows how Corollary III.0.3 allows us to recover some of our results in [8]. Of course, in [8] we also gave a precise description of the field generated by the Taylor coefficients of ϕ. We will leave it to the reader to use Corollary III.0.3(b) to rederive a weak form of the formula we found for the multiplicative reduction example studied in [8, I.0.3].

1. A Reduction Lemma

In this section we will show that it suffices to prove Theorem III.0.1 under the assumption that the elliptic surface $E \to C$ is semi-stable. We will make frequent use of the terminology and results on Weil local height functions described in [5, Chap. 10], which we will assume the reader is familiar with. We begin by fixing some height functions.

For every point $\gamma \in C$ we choose a function $\xi_\gamma \in \bar{K}(C)$ such that ξ_γ has a zero of order $g+1$ at γ and no other zeros. Such a function exists by the Riemann–Roch theorem. Note then that the map

$$(t, v) \mapsto \log^+ |\xi_\gamma(t)^{-1}|_v$$

is a Weil local height function relative to the divisor $(g+1)(\gamma)$. During the course of proving Theorem III.0.1 we will use the particular Weil local height functions defined by

$$\begin{aligned} \lambda_{C,D}\colon (C\setminus|D|) \times M &\to \mathbb{R} \\ (t; v) &\mapsto \frac{1}{g+1} \sum_{\gamma \in C} (\operatorname{ord}_\gamma D) \log^+ |\xi_\gamma(t)^{-1}|_v. \end{aligned} \tag{6}$$

We mention that (6) is a good definition for non-archimedean v, which is all we require for Theorem III.0.1, but for Theorem III.0.2 we will need a better definition for archimedean v as described in Section 3.

We begin with an elementary separation lemma that will be used in the proof of Theorem III.0.1.

LEMMA III.1.1. *Let $\gamma, \gamma' \in C$ be distinct points. There is a finite set of places $S \subset M_K$ such that*

$$\{(t, v) \in C \times (M \backslash S) : |\xi_\gamma(t)|_v < 1 \text{ and } |\xi_{\gamma'}(t)|_v < 1\} = \varnothing.$$

Proof. The function

$$(t, v) \mapsto \max\{|\xi_\gamma(t)|_v, |\xi_{\gamma'}(t)|_v\}$$

is M_K-bounded below, since the functions ξ_γ and $\xi_{\gamma'}$ have no common zeros. Hence for all but finitely many places $v \in M_K$,

$$\max\{|\xi_\gamma(t)|_v, |\xi_{\gamma'}(t)|_v\} \geqslant 1 \qquad \text{for all} \quad t \in C. \quad \blacksquare$$

Next we show that it suffices to prove Theorem III.0.1 for semi-stable elliptic surfaces.

REDUCTION LEMMA III.1.2. *If Theorem* III.0.1 *is true for elliptic surfaces $E \to C$ with semi-stable reduction, then it is true for all elliptic surfaces.*

Proof. First we note that any two Weil local heights for a given divisor D differ by an M_K-bounded function, so in particular they are identical for all but finitely many v in M_K. Hence it suffices to prove Theorem III.0.1 using the Weil heights defined by (6).

Let $E \to C$ be any elliptic surface with non-zero section $P: C \to E$. Let $\mu: C' \to C$ be a finite map of smooth projective curves, let E' be a minimal model for the pullback $E \times_C C'$, and let $P': C' \to E'$ be the extension of the section P. We take such a C' so that everything is defined over K and so that E' has semi-stable reduction, and then our assumption is that Theorem III.0.1 is true for E' and P'. This means we can find a finite set $S' \subset M_K$ such that

$$\begin{aligned} &\hat{\lambda}_{E'_{t'}}(P'_{t'}; v) - \lambda_{C', \hat{\eta}_{E'}(P')}(t'; v) = 0 \\ &\qquad \text{for all} \quad t' \in C' \backslash |\hat{\eta}_{E'}(P')| \text{ and all } v \in M_K \backslash S'. \end{aligned} \tag{7}$$

By the invariance of the Néron local height we have

$$\hat{\lambda}_{E_{\mu(t')}}(P_{\mu(t')}; v) = \hat{\lambda}_{E'_{t'}}(P'_{t'}; v) \qquad \text{for all} \quad (t', v) \in C' \times M. \tag{8}$$

Also

$$e_{\gamma'}(\mu)\, \hat{\lambda}_E(P; \operatorname{ord}_{\mu(\gamma')}) = \hat{\lambda}_{E'}(P'; \operatorname{ord}_{\gamma'}) \qquad \text{for all} \quad \gamma' \in C',$$

where $e_{\gamma'}(\mu)$ is the ramification index of μ at γ'. The reason this ramification index appears is that $\mathrm{ord}_{\gamma'}$ and $\mathrm{ord}_{\mu(\gamma')}$ are normalized to satisfy $\mathrm{ord}_{\gamma'} \circ \mu^* = e_{\gamma'}(\mu)\,\mathrm{ord}_{\mu(\gamma')}$. Summing over $\gamma' \in C'$ gives

$$\begin{aligned}
\hat{\eta}_{E'}(P') &= \sum_{\gamma' \in C'} \hat{\lambda}_{E'}(P'; \mathrm{ord}_{\gamma'}) \cdot (\gamma') \\
&= \sum_{\gamma' \in C'} e_{\gamma'}(\mu)\, \hat{\lambda}_E(P; \mathrm{ord}_{\mu(\gamma')}) \cdot (\gamma') \\
&= \sum_{\gamma \in C} \hat{\lambda}_E(P; \mathrm{ord}_\gamma) \sum_{\gamma' \in \mu^{-1}(\gamma)} e_{\gamma'}(\mu) \cdot (\gamma') \\
&= \sum_{\gamma \in C} \hat{\lambda}(P; \mathrm{ord}_\gamma) \cdot \mu^*((\gamma)) \\
&= \mu^* \hat{\eta}_E(P).
\end{aligned} \tag{9}$$

Hence by functoriality of local heights,

$$\lambda_{C', \hat{\eta}_{E'}(P')} = \lambda_{C', \mu^* \hat{\eta}_E(P)} = \lambda_{C, \hat{\eta}_E(P)} \circ \mu \qquad (\text{mod } M_K\text{-bounded functions}).$$

Assume now that we have fixed particular normalized Weil local height functions for each of the divisors $\hat{\eta}_{E'}(P')$ and $\eta_E(P)$ as described at the beginning of this section. Recall also that an M_K-bounded function is identically zero for all but finitely many places of K. Hence we can find a finite set of places $S'' \subset M_K$ so that

$$\lambda_{C', \hat{\eta}_{E'}(P')}(t'; v) = \lambda_{C, \hat{\eta}_E(P)}(\mu(t'); v) \qquad \text{for all} \quad (t', v) \in C' \times (M \backslash S''). \tag{10}$$

Combining (7), (8), and (10), and letting $S = S' \cup S''$, we find that

$$\hat{\lambda}_{E_{\mu(t')}}(P_{\mu(t')}; v) - \lambda_{C, \hat{\eta}_E(P)}(\mu(t'); v) = 0 \qquad \text{for all} \quad t' \in C' \backslash |\hat{\eta}_{E'}(P')| \text{ and all } v \in M \backslash S.$$

But $\mu: C' \to C$ is surjective (note we are working over $\bar{K}$) and from (9) we have $|\hat{\eta}_E(P)| = \mu_* |\hat{\eta}_{E'}(P')|$. This gives the desired result

$$\hat{\lambda}_{E_t}(P_t; v) - \lambda_{C, \hat{\eta}_E(P)}(t; v) = 0 \qquad \text{for all} \quad t \in C \backslash |\hat{\eta}_E(P)| \text{ and all } v \in M \backslash S. \quad \blacksquare$$

2. Proof of Theorem III.0.1

In this section we will give the proof of Theorem III.0.1. Using Reduction Lemma III.1.2, we may assume that the elliptic surface $E \to C$ is semi-stable. That is, it is a minimal surface all of whose fibers are either smooth or of multiplicative type. Further, just as in the proof of Reduction Lemma III.1.2, it suffices to prove Theorem III.0.1 for the particular choice of Weil local height functions described in (6).

Let

$$\{\gamma_1, ..., \gamma_n\} = \{\gamma \in C : E_\gamma \text{ is singular or } P_\gamma = \mathcal{O}_\gamma\}.$$

Then we can write

$$\hat{\eta}_E(P) = \sum_{i=1}^{n} \hat{\lambda}_E(P; \operatorname{ord}_{\gamma_i}) \cdot (\gamma_i),$$

since the explicit formulas for the local height show that $\hat{\eta}_E(P)$ is supported on the set $\{\gamma_1, ..., \gamma_n\}$. In fact, since $E \to C$ is semi-stable, we also have $\hat{\lambda}_E(P; \operatorname{ord}_{\gamma_i}) \neq 0$ for each $1 \leqslant i \leqslant n$. To ease notation, we will set

$$\xi_i = \xi_{\gamma_i} \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

and we will let

$$C_0 = C \setminus \{\gamma_1, ..., \gamma_n\}.$$

For later use, we will also fix some point $\gamma_0 \in C$ distinct from the other γ_i's, and we will set ξ_0 to be the constant function 1. Define a function $F_P : C_0 \times M \to \mathbb{R}$ by the formula

$$F_P(t, v) = \hat{\lambda}_{E_t}(P_t; v) - \lambda_{C, \hat{\eta}_E(P)}(t; v).$$

We need to prove that there is a finite set of places $S \subset M_K$ such that $F_P = 0$ on $C_0 \times (M \setminus S)$.

Let $U \subset C$ be any affine subset defined over K such that E has a minimal Weierstrass equation over U and with $\{\gamma_1, ..., \gamma_n\} \subset U$. Fix generators $u_1, ..., u_m \in K(C)$ for the ring of regular functions on U, so

$$K[U] = K[u_1, ..., u_m].$$

Given these data, define a set

$$V = \{(t, v) \in C \times M : |u_i(t)|_v \leqslant 1 \text{ for all } 1 \leqslant i \leqslant m\}.$$

In the terminology of [5, Chap. 10], V is an affine M_K-bounded set subordinated to U. For any set of places $S \subset M_K$, we let V^S be the part of V that is in C_0 and is "disjoint" from S:

$$V^S = V \cap (C_0 \times (M \setminus S)).$$

We are going to prove:

There is a finite set of places $S \subset M_K$ such that $F_P(t, v) = 0$ for all $(t, v) \in V^S$. $(*)$

Assuming $(*)$, it is easy to complete the proof of Theorem III.0.1 as follows. From [5, Chap.10, Section 1] we see that it is possible to cover $C \times M$ with finitely many such V's, say

$$C \times M = V_1 \cup \cdots \cup V_r.$$

Then, letting S_i be the set of places corresponding to V_i in $(*)$ and setting $S = S_1 \cup \cdots \cup S_r$, we get the desired result that $F_P = 0$ on $C_0 \times (M \backslash S)$. We are thus reduced to proving that $(*)$ is true.

We start by letting S be the finite set of places

$$S = M_K^\infty \cup \{v \in M_k : v(6) > 0\}.$$

As we proceed with the proof we will enlarge S as necessary, always with the requirement that S be finite. For each $1 \leqslant i \leqslant n$ define

$$V_i = \{(t, v) \in V : |\xi_i(t)|_v < 1\},$$

and let V_0 be the rest of V,

$$V_0 = V \Big\backslash \left(\bigcup_{i=1}^{n} V_i \right) = \{(t, v) \in V : |\xi_i(t)|_v \geqslant 1 \text{ for all } 1 \leqslant i \leqslant n\}.$$

Further, let

$$V_i^S = V_i \cap V^S \qquad \text{for} \quad 0 \leqslant i \leqslant n.$$

Intuitively, for $1 \leqslant i \leqslant n$ the set V_i consists of pairs (t, v) such that t is v-adically close to γ_i, while V_0 consists of pairs (t, v) such that t is v-adically far away from all of $\gamma_1, \ldots, \gamma_n$. Since V^0 is the union of the V_i^Ss, in order to prove $(*)$, it suffices to find an S so that $F_P = 0$ on each of the V_i^Ss.

We begin by applying Lemma III.1.1 to each distinct pair of points γ_i, γ_j. Enlarging S, this allows us to assume

$$V_i^S \cap V_j^S = \varnothing \qquad \text{for all} \quad 0 \leqslant i < j \leqslant n.$$

This disjointedness will make the proof of $(*)$ notationally much simpler.

Next we take a minimal Weierstrass equation for E/U of the form

$$E: y^2 = x^3 + Ax + B, \qquad A, B \in K[U], \qquad \Delta_E = 4A^3 + 27B^2 \neq 0. \quad (11)$$

The assumption that E is semi-stable tells us that A and B have no common zeros in U, so by the Nullstellensatz we can write

$$AA' + BB' = 1 \qquad \text{for some} \quad A', B' \in K[U]. \tag{12}$$

Making a further enlargement of S, we may assume that

$$A, B, A', B' \in R_S[U] = R_S[u_1, ..., u_m],$$

where R_S is the ring of S-integers of K. Then for $(t, v) \in V^S$ the (non-archimedean) triangle inequality says that

$$|A(t)|_v, |B(t)|_v, |A'(t)|_v, |B'(t)|_v \leqslant 1,$$

so (12) implies that

$$\max\{|A(t)|_v, |B(t)|_v\} = 1 \qquad \text{for all} \quad (t, v) \in V^S.$$

This in turn implies that for all $(t, v) \in V^S$ the Weierstrass equation

$$E_t\colon y^2 = x^3 + A(t)\, x + B(t)$$

is minimal at v and has semi-stable (i.e., good or multiplicative) reduction.

We next write the divisor of the function $\Delta_E \in K(C)$ as

$$\operatorname{div}(\Delta_E) = \sum_{i=1}^{n} (\operatorname{ord}_{\gamma_i} \Delta_E) \cdot (\gamma_i) + D_1 \qquad \text{with} \quad |D_1| \subset C \backslash U.$$

We can do this since the zeros of Δ_E on U are exactly the point $\gamma \in U$ such that the fiber E_γ is singular. Note that, since D_1 is supported on $C \backslash U$ and V is affine M_K-bounded subordinated to U, we have

$$\lambda_{C, D_1} \qquad \text{is } M_K\text{-bounded on } V.$$

Hence, expanding S yet again, we get $\lambda_{C, D_1} = 0$ on V^S.

By functoriality of local heights we have

$$\begin{aligned} v(\Delta_E(t)) = \sum_{i=1}^{n} (\operatorname{ord}_{\gamma_i} \Delta_E)\, \lambda_{C,(\gamma_i)}(t; v) + \lambda_{C, D_1}(t; v) \\ + (\text{an } M_K\text{-bounded function}) \qquad \text{on } C_0 \times M. \end{aligned}$$

But an M_K-bounded function is identically zero for all but finitely many $v \in M_K$, so enlarging S yet again we find that

$$v(\Delta_E(t)) = \sum_{i=1}^{n} (\operatorname{ord}_{\gamma_i} \Delta_E)\, \lambda_{C,(\gamma_i)}(t; v) \qquad \text{for all} \quad (t, v) \in V^S.$$

Recall we are using the Weil local height functions $\lambda_{C,(\gamma_i)}$ defined by (6). Combining this with the definition of the V_i^Ss, we find for each $0 \leqslant i \leqslant n$ that

$$v(\Delta_E(t)) = \frac{1}{g+1}(\mathrm{ord}_{\gamma_i} \Delta_E) \log^+ |\xi_i(t)^{-1}|_v \qquad \text{for all } (t, v) \in V_i^S. \quad (13)$$

Note that for $i = 0$ we have set $\xi_0 = 1$, so the right-hand side is 0, as it should be.

We are finally ready to look at the section $P: C \to E$. Using the Weierstrass equation (11) we write

$$P = (X_P, Y_P) \qquad \text{with} \quad X_P, Y_P \in K(C).$$

Note that for $\gamma \in U$ we have $P_\gamma = \mathcal{O}_\gamma$ if and only if X_P has a pole at γ. This means that

$$\mathrm{div}(X_P) = \sum_{i=1}^{n} (\mathrm{ord}_{\gamma_i}^+ X_P^{-1}) \cdot (\gamma_i) + D_2 - D_3$$
$$\text{with } |D_2| \subset C \backslash U \text{ and } D_2, D_3 \geqslant 0.$$

Hence

$$\begin{aligned} v^+(X_P(t)^{-1}) &= \max\{v(X_P(t)^{-1}), 0\} \\ &= \sum_{i=1}^{n} (\mathrm{ord}_{\gamma_i}^+ X_P^{-1}) \lambda_{C,(\gamma_i)}(t; v) + \lambda_{C,D_2}(t; v) \\ &\quad + (\text{an } M_K\text{-bounded function}) \qquad \text{on } C_0 \times M. \end{aligned}$$

As above, we can enlarge S to make λ_{C,D_2} and the M_K-bounded function vanish on V^S. This leads to

$$v^+(X_P(t)^{-1}) = \frac{1}{g+1}(\mathrm{ord}_{\gamma_i}^+ X_P^{-1}) \log^+ |\xi_i(t)^{-1}|_v$$
$$\text{for all} \quad (t, v) \in V_i^S. \quad (14)$$

Using the same argument for a third and fourth time, we get after enlarging S yet again that

$$\begin{aligned} &\min\{v(Y_P(t)), v(3X_P(t)^2 + A(t))\} \\ &\quad = \frac{1}{g+1} \min\{\mathrm{ord}_{\gamma_i}(Y_P), \mathrm{ord}_{\gamma_i}(3X_P^2 + A)\} \\ &\quad \times \log^+ |\xi_i(t)^{-1}|_v \qquad \text{for all} \quad (t, v) \in V_i^S, \text{ and} \end{aligned} \quad (15)$$

$$v^+(Y_P(t)) = \frac{1}{g+1}(\mathrm{ord}_{\gamma_i}^+ Y_P) \log^+ |\xi_i(t)^{-1}|_v \qquad \text{for all} \quad (t, v) \in V_i^S. \quad (16)$$

To see that these hold even for $i=0$, we note that if $Y_P(t)=3X_P(t)^2+A(t)=0$ for some $t\in U$, then necessarily the fiber E_t is singular, so $t=\gamma_i$ for some $1\leqslant i\leqslant n$.

We also observe that the disjointedness of the V_i^Ss and our choice of Weil local heights gives

$$\begin{aligned}\lambda_{C,\hat{\eta}_E(P)}(t;v)&=\frac{1}{g+1}\sum_{i=1}^{n}\hat{\lambda}_E(P;\mathrm{ord}_{\gamma_i})\log^+|\xi_i(t)^{-1}|_v\\&=\frac{1}{g+1}\hat{\lambda}_E(P;\mathrm{ord}_{\gamma_i})\log^+|\xi_i(t)^{-1}|_v\qquad\text{for}\quad (t,v)\in V_i^S.\end{aligned}\tag{17}$$

And we reiterate that, for all $(t,v)\in V^S$, the Weierstrass equation

$$E_t\colon y^2=x^3+A(t)x+B(t)\tag{18}$$

is minimal at v.

Now let $(t,v)\in V_i^S$. We are going to show that $F_P(t,v)=0$ by considering two cases.

Case I. $\min\{v(Y_P(t)),v(3X_P(t)^2+A(t))\}\leqslant 0$.

This is the case that $P_t\in E_t^0(\bar{K}_v)$; that is, P_t is on the identity component of the special fiber of E_t at v. The v-adic local height is then given by the formula

$$\hat{\lambda}_{E_t}(P_t;v)=\tfrac{1}{2}v^+(X_P(t)^{-1})+\tfrac{1}{12}v(\Delta_E(t)).$$

Note we are using the fact that (18) is minimal at v. From (13) and (14) we get

$$\hat{\lambda}_{E_t}(P_t;v)=\left\{\frac{1}{2}\mathrm{ord}_{\gamma_i}^+(X_P^{-1})+\frac{1}{12}\mathrm{ord}_{\gamma_i}(\Delta_E)\right\}\cdot\frac{1}{g+1}\log^+|\xi_i(t)^{-1}|_v.\tag{19}$$

Further, (15) and the assumption for Case I imply that

$$\min\{\mathrm{ord}_{\gamma_i}(Y_P),\mathrm{ord}_{\gamma_i}(3X_P^2+A)\}\leqslant 0.$$

(For $1\leqslant i\leqslant n$ this is immediate from (15). For $i=0$ it follows from the fact that E_{γ_0} is smooth.) This means that the section P goes through the identity component of the fiber E_{γ_i}, so

$$\hat{\lambda}_E(P;\mathrm{ord}_{\gamma_i})=\tfrac{1}{2}\mathrm{ord}_{\gamma_i}^+(X_P^{-1})+\tfrac{1}{12}\mathrm{ord}_{\gamma_i}(\Delta_E).\tag{20}$$

Combining (19), (20), and (17) gives

$$F_P(t,v)=\hat{\lambda}_{E_t}(P_t;v)-\lambda_{C,\hat{\eta}_E(P)}(t;v)=0.$$

This completes the proof of $(*)$ for Case I.

Case II. $\min\{v(Y_P(t)), v(3X_P(t)^2+A(t))\}>0$.

This is the case that $P_t \notin E_t^0(\bar{K}_v)$; that is, P_t is not on the identity component of the special fiber of E_t at v. Note that we automatically have $v(\Delta_E(t))>0$. The v-adic local height in this case is given by the formula [7]

$$\hat{\lambda}_{E_t}(P_t; v)=\frac{1}{2}B_2\left(\frac{v(Y_P(t))}{v(\Delta_E(t))}\right)v(\Delta_E(t)),$$

where $B_2(Z)=Z^2-Z+\frac{1}{6}$ is the 2nd Bernoulli polynomial. Substituting (16) and (13) into this formula gives

$$\hat{\lambda}_{E_t}(P_t; v)=\left\{\frac{1}{2}B_2\left(\frac{\operatorname{ord}_{\gamma_i} Y_P}{\operatorname{ord}_{\gamma_i}\Delta_E}\right)\operatorname{ord}_{\gamma_i}\Delta_E\right\}\cdot\frac{1}{g+1}\log^+|\xi_i(t)^{-1}|_v. \tag{21}$$

Note how the $\log^+|\xi_i(t)^{-1}|_v$s cancel out in the numerator and the denominator of the argument of B_2.

Using (15) and the assumption we have made for Case II gives

$$\min\{\operatorname{ord}_{\gamma_i}(Y_P), \operatorname{ord}_{\gamma_i}(3X_P^2+A)\}>0,$$

and also $|\xi_i(t)|_v<1$, so $i\neq 0$. Hence P_i does not go through the identity component of the fiber E_{γ_i}, which means that

$$\hat{\lambda}_E(P; \operatorname{ord}_{\gamma_i})=\frac{1}{2}B_2\left(\frac{\operatorname{ord}_{\gamma_i} Y_P}{\operatorname{ord}_{\gamma_i}\Delta_E}\right)\operatorname{ord}_{\gamma_i}\Delta_E. \tag{22}$$

Now (21), (22), and (17) give $F_P(t, v)=0$ in this case.

Combining Cases I and II, we have shown that $F_P(t, v)=0$ for all $(t, v)\in V_i^S$. This holds for each $0\leqslant i\leqslant n$, so $F_P(t, v)=0$ for $(t, v)\in V^S$. This completes the proof of $(*)$, which as noted earlier also completes the proof of Theorem III.0.1. ∎

3. Analytic Weil Heights

In this section we will define and give examples of analytic Weil heights. These are similar to the Weil heights described in [5] except that the property of being M_K-continuous is replaced by a stronger analyticity condition. At the end of this section we will also describe the set of functions $\mathscr{F}_{E,S}$ which is used in the statement of Theorem III.0.2.

Let $D \in \mathrm{Div}(C) \otimes \mathbb{Q}$ be a divisor, let $v \in M$ be a place of $\bar{K}$, and let $\phi \colon C(\mathbb{C}_v) \setminus |D| \to \mathbb{R}$ be a function. We consider the following properties that ϕ might possess:

(i) If v is $\begin{pmatrix}\text{archimedean}\\ \text{non-archimedean}\end{pmatrix}$, then ϕ is $\begin{pmatrix}\text{real analytic}\\ \text{locally constant}\end{pmatrix}$ on $C(\mathbb{C}_v) \setminus |D|$.

(ii) Let $\gamma \in |D|$, and let $u \in \mathbb{C}_v(C)$ be a rational function that vanishes at γ. If v is $\begin{pmatrix}\text{archimedean}\\ \text{non-archimedean}\end{pmatrix}$, then the difference

$$\phi(t) - \frac{\mathrm{ord}_\gamma D}{\mathrm{ord}_\gamma u} \log |u(t)^{-1}|_v$$

extends to a $\begin{pmatrix}\text{real analytic}\\ \text{locally constant}\end{pmatrix}$ function in a v-adic neighborhood of γ.

DEFINITION. Let $D \in \mathrm{Div}(C) \otimes \mathbb{Q}$. An *analytic Weil local height function corresponding to the divisor D* is a Weil local height function for D,

$$\lambda_{C,D} \colon (C \setminus |D|) \times M \to \mathbb{R},$$

such that for any fixed $v \in M$, the function $t \to \lambda_{C,D}(t; v)$ satisfies the two properties (i) and (ii) described above. For the definition of Weil local height functions, see [5, Chap. 10, Section 2]. In essence, we have replaced the continuity condition in [5] with the more stringent analyticity conditions (i) and (ii).

For each finite extension L/K we normalize the absolute values in M_L in the usual way (see [5]), and for $w \in M_L$ we let

$$n_w = \frac{[L_w : K_w]}{[L : K]}.$$

DEFINITION. An *analytic Weil (global) height function corresponding to the divisor D* is a function

$$h_{C,D} \colon C(\bar{K}) \to \mathbb{R}$$
$$t \mapsto \sum_{w \in M_L} n_w \lambda_{C,D}(t; w) \qquad \text{for} \quad t \in C(L),$$

where the $\lambda_{C,D}$s are analytic Weil local heights for D. One checks in the usual way that $h_{C,D}(t)$ is independent of the choice of a field of definition for t.

Actually, $h_{C,D}$ is not well-defined for points in the support of D. For our purposes it will suffice to make the ad hoc definition that

$$h_{C,D}(t)=0 \qquad \text{for all} \quad t\in |D|.$$

In general, there may be a more intrinsic way to define $h_{C,D}(t)$ for $t\in|D|$ which yields non-zero values. We will describe one possibility in Example 1(b) below.

Remark. Recall that the standard height function on $\mathbb{P}^1(\mathbb{Q})$ with respect to the divisor (∞) is defined by

$$h(t)=\log^+|t|_\infty+\sum_p \log^+|t|_p.$$

This height is not an analytic Weil height. The problem is at the archimedean place, since the function $\log^+|x|_\infty$ is not real analytic in the neighborhood of a point x_0 with $|x_0|_\infty=1$. We can correct the problem by using instead

$$\tfrac{1}{2}\log(1+t^2)+\sum_p \log^+|t|_p.$$

We now generalize this example to construct analytic Weil heights for any curve and any divisor.

Example 1(a). As in Section 1, we start by fixing for each point $\gamma\in C$ a function $\xi_\gamma\in\bar{K}(C)$ such that ξ_γ has a zero of order $g+1$ at γ and no other zeros. The map

$$(t,v)\mapsto \log^+|\xi_\gamma(t)^{-1}|_v$$

is a Weil local height function relative to the divisor $(g+1)(\gamma)$, but for archimedean absolute values it will not be analytic around points with $|\xi_\gamma(t)|_v=1$. So at archimedean absolute values we use instead

$$(t,v)\mapsto \tfrac{1}{2}\log(1+|\xi_\gamma(t)^{-1}|_v^2).$$

Now for any divisor $D\in\operatorname{Div}(C)\otimes\mathbb{Q}$ we sum these to get

$$\lambda_{C,D}\colon (C\backslash|D|)\times M\to\mathbb{R}$$

$$\lambda_{C,D}(t;v)=\begin{cases}\dfrac{1}{g+1}\displaystyle\sum_{\gamma\in C}(\operatorname{ord}_\gamma D)\log^+|\xi_\gamma(t)^{-1}|_v & \text{if} \quad v\in M^0,\\[2ex] \dfrac{1}{2(g+1)}\displaystyle\sum_{\gamma\in C}(\operatorname{ord}_\gamma D)\log(1+|\xi_\gamma(t)^{-1}|_v^2) & \text{if} \quad v\in M^\infty.\end{cases}$$

Then $\lambda_{C,D}$ is an analytic local Weil height function for the divisor D.

EXAMPLE 1(b). Consider the global Weil height $h_{C,D}$ defined by summing the local heights described in Example 1(a). We are going to compute $h_{C,D}$ for the special case that $D=(\gamma)$. Let $t\in C(L)$ with $t\neq\gamma$, and write the fractional ideal generated by $\xi_\gamma(t)$ as a quotient of relatively prime integral ideals,

$$(\xi_\gamma(t))=\mathfrak{a}\cdot\mathfrak{b}^{-1}.$$

Let h be the class number of L and write $\mathfrak{a}^h=(\alpha)$ and $\mathfrak{b}^h=(\beta)$. Adjusting α by a unit if necessary, we may assume that $\xi_\gamma(t)^h=(\alpha/\beta)$. Note that $\alpha\neq 0$, since $t\neq\gamma$ and γ is the only zero of ξ_γ. Then

$$\begin{aligned}(g+1)\,h_{C,(\gamma)}(t)&=\frac{1}{2}\sum_{w\in M_L^\infty} n_w\log(1+|\xi_\gamma(t)^{-1}|_w^2)\\&\quad+\sum_{w\in M_L^0} n_w\log^+|\xi_\gamma(t)^{-1}|_w\\&=\frac{1}{2}\sum_{w\in M_L^\infty} n_w\log\left(1+\left|\frac{\beta}{\alpha}\right|_w^{2/h}\right)\\&\quad+\sum_{w\in M_L^0} n_w\log^+\left|\frac{\beta}{\alpha}\right|_w^{1/h}\\&=\frac{1}{2}\sum_{w\in M_L^\infty} n_w\log\left(1+\left|\frac{\beta}{\alpha}\right|_w^{2/h}\right)\\&\quad+\sum_{w\in M_L^\infty} n_w\log|\alpha|_w^{1/h}\qquad\text{from the product formula}\\&=\frac{1}{2}\sum_{w\in M_L^\infty} n_w\log(|\alpha|_w^{2/h}+|\beta|_w^{2/h}).\end{aligned}$$

In other words, for any $t\neq\gamma$ we write $\xi_\gamma(t)^h=\alpha/\beta$ with α and β relatively prime, and then

$$h_{C,(\gamma)}(t)=\frac{1}{2(g+1)}\sum_{w\in M_L^\infty} n_w\log(|\alpha|_w^{2/h}+|\beta|_w^{2/h}).$$

But this last formula makes sense even for $t=\gamma$, since in that case we have $\xi_\gamma(\gamma)=0$, so $\alpha=0$ and $\beta=1$. So at least for the particular local heights described in Example 1(a), it is reasonable to define the corresponding global height to satisfy

$$h_{C,(\gamma)}(\gamma)=0.$$

Finally, this suggests that for an arbitrary divisor D we should define

$$h_{C,D}(t) = h_{C,D'_t}(t) \qquad \text{for} \quad t \in |D|, \text{ where } D'_t = D - (\operatorname{ord}_t D)(t).$$

EXAMPLE 2. Our second example of analytic height function is less elementary, but more intrinsic. Let $D \in \operatorname{Div}(C) \otimes \mathbb{Q}$ be a divisor. Then for each place v we set

$$\lambda_{C,D}(t; v) = [D, t]_v,$$

where $[D, t]_v$ is the Arakelov intersection index of D and t at v. Thus for archimedean v it is the value at t of a Green's function for D, and for non-archimedean v it is the intersection of a thickening of D and t on the special fiber of a minimal model for C over Spec R_v. For $v \in M^\infty$, we see that λ is an analytc height because the Green's function is analytic away from $|D|$ and has a logarithmic singularity along $|D|$. For $v \in M^0$, it is an easy exercise using the classical definition of intersection index to see that λ has the required properties.

For a general discussion of the relationship between Green's functions and heights, the reader might consult [5, Chapter 13], [2], or [4]. We also remark that if we take $C = \mathbb{P}^1$, $D = (\infty)$, and $\xi_\gamma(t) = t - \gamma$ in Example 1(a), and if we use the Fubini–Study metric on $\mathbb{P}^1$ to define our Green's function, then the local heights in Examples 1(a) and 2 will be the same. See [2, page 291].

One final task remains before we commence the proof of Theorem III.0.2. We need to define the set of functions $\mathscr{F}_{E,S}$. This will require several preliminary definitions. We fix some Weierstrass equation for E/C defined over K and let c_4 and c_6 be the usual quantities (see [6]). For each point $t_0 \in C(\bar{K})$ let $e(t_0)$ be the integer

$$e(t_0) = \text{Denominator}\left(\frac{\min\{\operatorname{ord}_{t_0} c_4^3, \operatorname{ord}_{t_0} c_6^2\}}{12}\right).$$

We remark that $e(t_0) \in \{1, 2, 3, 4, 6\}$, that $e(t_0)$ is independent of the choice of Weierstrass equation, and that $e(t_0) = 1$ if and only if E has semi-stable (i.e., good or multiplicative) reduction over t_0.

Next let $e \geqslant 1$ be an integer and $v \in M$. Define a ring of functions $\mathscr{R}_{e,v}$ as follows. If v is non-archimedean, then $F: \mathbb{C}_v \to \mathbb{R}$ is in $\mathscr{R}_{e,v}$ if it is a constant function. If v is archimedean, then $F: \mathbb{C}_v \to \mathbb{R}$ is in $\mathscr{R}_{e,v}$ if it has the form

$$F(z) = \sum_{i=0}^{e-1} f_i(z)\, |z|_v^{2/e},$$

where $f_0, \ldots, f_{e-1}$ are real analytic in a neighborhood of 0. Note $\mathscr{R}_{e,v}$ is a ring, since $|z|^2 = x^2 + y^2$ is real analytic at 0. Also, if $e = 1$ and v is archimedean, then $\mathscr{R}_{e,v}$ is just the ring of functions analytic in a neighborhood of 0.

Again let $v \in M$, and let

$$C_0 = \{t \in C \colon E_t \text{ is smooth}\}.$$

We define a set of functions $\mathscr{F}_{E,v}$ to consist of functions $\phi\colon C(\mathbb{C}_v) \to \mathbb{R}$ with the following properties:

(i) If v is archimedean, then ϕ is real analytic on $C_0(\mathbb{C}_v)$.

(ii) If v is non-archimedean, then ϕ is locally constant on $C_0(\mathbb{C}_v)$.

(iii) Let $t_0 \in (C \backslash C_0)(\mathbb{C}_v)$ and let u be a uniformizer at t_0. Then there are functions $F_1, F_2, F_3 \in \mathscr{R}_{e(t_0),v}$ such that

$$\phi(t) = F_1(u(t)) + \frac{F_2(u(t))}{\log|u(t)|_v + F_3(u(t))}$$

for all $t \in C(\mathbb{C}_v)$ in a v-adic neighborhood of t_0.

(iv) In (iii), if $\operatorname{ord}_{t_0} j_E \geqslant 0$, then $F_2 = 0$. Equivalently, $\phi(t) = F(u(t))$ for some $F \in \mathscr{R}_{e(t_0),v}$.

With these preliminaries, we can now define $\mathscr{F}_{E,S}$. Recall that S is a finite subset of M_K. For each $v \in S$ and for any finite extension L/K we have several embeddings $C(L) \subsetneq C(\mathbb{C}_v)$, essentially one for each $w \in M_L$ with $w \mid v$. If $t \in C(L)$, we will denote the corresponding image of t in $C(\mathbb{C}_v)$ by t_w. Then $\mathscr{F}_{E,S}$ is the set of functions $\phi\colon C(\bar{K}) \to \mathbb{R}$ of the form

$$\phi(t) = \sum_{v \in S} \sum_{\substack{w \in M_L \\ w \mid v}} n_w \phi_v(t_w),$$

where $\phi_v \in \mathscr{F}_{E,v}$. Note that, for any given $t \in C(\bar{K})$, we are choosing a finite extension L/K with $t \in C(L)$. One checks in the usual way that $\phi(t)$ is independent of the choice of L.

4. Proof of Theorem III.0.2

In this section we will prove Theorem III.0.2. More precisely, we will prove a local version of Theorem 0.2, and then adding this local theorem over all v will immediately give the desired global result.

THEOREM III.4.1. *Let $E \to C$ and $P: C \to E$ be as defined in the introduction, let $v \in M_K$ be a place which we extend to $\bar{K}$ in some fashion, and let $\lambda_{C,\hat{\eta}_E(P)}$ be an analytic Weil local height for the divisor $\hat{\eta}_E(P)$. Then there is a function $\phi_{P,v} \in \mathscr{F}_{E,v}$ so that*

$$\hat{\lambda}_{E_t}(P_t; v) = \lambda_{C,\hat{\eta}_E(P)}(t; v) + \phi_{P,v}(t) \qquad \textit{for all} \quad t \in C(\bar{K}_v)\setminus|\hat{\eta}_E(P)|. \tag{23}$$

Further, for all but finitely many $v \in M_K$, the function $\phi_{P,v}$ is identically zero. (See Section 3 for the definition of analytic Weil local height and a description of the set $\mathscr{F}_{E,v}$.)

Proof. First, let $S \subset M_K$ be the finite set of places described in Theorem III.0.1. Then for all $v \notin S$, Theorem III.0.1 says that (23) is true with $\phi_{P,v} = 0$. This proves the second part of Theorem III.4.1, so we are left to consider (23) for a fixed $v \in M_K$.

Next we observe that the functions in $\mathscr{F}_{E,v}$ are characterized by properties that are local for the v-adic topology, since they are described in terms of functions that are locally constant (non-archimedean case) or real analytic (archimedean case). Thus it suffices to verify (23) in any neighborhood U of a given point $t_0 \in C(\bar{K})$. In other words, (23) is really saying that for any point $t_0 \in C(\bar{K})$, the difference

$$\hat{\lambda}_{E_t}(P_t; v) - \lambda_{C,\hat{\eta}_E(P)}(t; v) \tag{24}$$

extends in a certain "nice" way to a function in a v-adic neighborhood of t_0.

Let u be a uniformizer at t_0. Then Theorem II.0.1 in [9] says precisely that the difference

$$\hat{\lambda}_{E_t}(P_t; v) - \hat{\lambda}_E(P; \mathrm{ord}_{t_0}) \log|u(t)^{-1}|_v \tag{25}$$

extends to a function in $\mathscr{F}_{E,v}$ in some neighborhood $U_1 \subset C(\mathbb{C}_v)$ of t_0. On the other hand, since

$$\mathrm{ord}_{t_0}(\hat{\eta}_E(P)) = \hat{\lambda}_E(P; \mathrm{ord}_{t_0}),$$

the definition of analytic Weil local height says that the difference

$$\lambda_{C,\hat{\eta}_E(P)}(t; v) - \hat{\lambda}_E(P; \mathrm{ord}_{t_0}) \log|u(t)^{-1}|_v \tag{26}$$

extends to a real analytic function (if v is archimedean) or locally constant function (if v is non-archimedean) in a neighborhood $U_2 \subset C(\mathbb{C}_v)$ of t_0.

Subtracting (26) from (25) tells us that (24) extends to a function in $\mathscr{F}_{E,v}$ on $U_1 \cap U_2$. Since this holds for all $t_0 \in C(\bar{K})$, we conclude that (24) extends to a function in $\mathscr{R}_{E,v}$, which completes the proof of Theorem III.4.1. ∎

We next prove Theorem III.0.2 by summing Theorem III.4.1 over v.

Proof (of Theorem III.0.2). For each $v \in M_K$ let $\phi_{P,v} \in \mathscr{R}_{E,v}$ be the function described in Theorem III.4.1, and let $S \subset M_K$ be the finite set of places for which $\phi_{P,v}$ is not identically zero.

Let $t \in C(L)$ for some finite extension L/K. For each $w \in M_L$ with $w \mid v$ we have an embedding $C(L) \hookrightarrow C(\bar{K}_v)$, and we denote the image of t by $t_w \in C(\bar{K}_v)$. Then by definition

$$\hat{h}_{E_t}(P_t) = \sum_{w \in M_L} n_w \hat{\lambda}_{E_t}(P_t; w),$$

$$h_{C,\hat{\eta}_E(P)}(t) = \sum_{w \in M_L} n_w \lambda_{C,\hat{\eta}_E(P)}(t; w).$$

Hence

$$\begin{aligned}
&\hat{h}_{E_t}(P_t) - h_{C,\hat{\eta}_E(P)}(t)\\
&\quad = \sum_{w \in M_L} n_w \{\hat{\lambda}_{E_t}(P_t; w) - \lambda_{C,\hat{\eta}_E(P)}(t; w)\}\\
&\quad = \sum_{v \in M_K} \sum_{\substack{w \in M_L\\ w \mid v}} n_w \phi_{P,v}(t_w) \qquad \text{from Theorem III.4.1,}\\
&\quad = \sum_{v \in S} \sum_{\substack{w \in M_L\\ w \mid v}} n_w \phi_{P,v}(t_w) \qquad \text{since } \phi_{P,v} = 0 \text{ for } v \notin S.
\end{aligned}$$

Referring to the definition of $\mathscr{F}_{E,S}$ in Section 3, we see that this last expression is in $\mathscr{F}_{E,S}$. This completes the proof of Theorem III.0.2. ∎

Using Theorem III.0.2, the proof of Corollary III.0.3 amounts to nothing more than unwinding the various definitions.

Proof (of Corollary III.0.3). We are given an elliptic surface $E \to \mathbb{P}^1$ and a section $P\colon \mathbb{P}^1 \to E$ all defined over $\mathbb{Q}$. Let $S \subset M_{\mathbb{Q}}$ and $F_P \in \mathscr{F}_{E,S}$ be the finite set of places and the function described in Theorem III.0.2. In particular, we have

$$\hat{h}_{E_t}(P_t) = h_{C,\hat{\eta}_E(P)}(t) + F_P(t) \tag{27}$$

for $t \in \mathbb{P}^1(\mathbb{Q})$ with E_t smooth and $P_t \neq \mathcal{O}_t$. Here F_P is a function of the form

$$F_P(t) = \sum_{p \in S} \phi_p(t), \tag{28}$$

where $\phi_p \in \mathscr{F}_{E,p}$ for each $p \in S$. Let

$$\gamma = (\gamma_p) \in \prod_{p \in S} \mathbb{P}^1(\mathbb{Q}_p)$$

be a given point.

(a) Suppose first that all of the fibers E_{γ_p} are smooth. Then by definition of $\mathscr{F}_{E,p}$, for each finite place $p \in S$ the function ϕ_p is constant in a p-adic neighborhood U_p of γ_p, and for the infinite place $p = \infty$ the function ϕ_∞ is real analytic in a p-adic neighborhood U_∞ of γ_∞. So if we define

$$U_\gamma = \prod_{p \in S} U_p,$$

then

$$\hat{h}_{E_t}(P_t) = h_{C, \bar{n}_E(P)}(t) + \phi_\infty(t) + (\text{constant})$$

for $t \in U_\gamma \cap \mathbb{P}^1(\mathbb{Q})$ with $P_t \neq \mathcal{O}_t$. We can absorb the constant into ϕ_∞, which completes the proof of (a).

(b) In general, the definition of $\mathscr{F}_{E,p}$ says that

$$\phi_p(t) = F_{p,1}(u_p(t)) + \frac{F_{p,2}(u_p(t))}{\log|u_p(t)|_p + F_{p,3}(u_p(t))}$$

in some p-adic neighborhood U_p of γ_p, where $F_1, F_2, F_3 \in \mathscr{R}_{e(\gamma_0),p}$. Note in particular that the $F_i \circ u_p$s are continuous maps from U_p to $\mathbb{R}$. This is true even if $p = \infty$ and/or $e(\gamma_0) > 1$. Hence there is a constant κ_p such that

$$\phi_p(t) = \kappa_p + O\left(\frac{1}{\log|u_p(t)|_p}\right) \qquad \text{for} \quad t \in U_p.$$

(More precisely, $\kappa_p = F_{p,1}(0)$.) Summing over $p \in S$ and using (27) and (28) gives the desired result. ∎

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